# On Some New Types of Totally Closed Functions

Amna Mohamed Abdelgader Ahmad

Department of Mathematics, Faculty of Sciences, Almergib University, AlKhums, Libya

**Abstract**

The aim of this paper is to introduce and study totally closed functions in topological spaces. Some weak and strong forms of this kind of functions are introduced and studied. Several results and properties are investigated and proven.

**الملخص:**

في هذا البحث، قدمنا دراسة لأنواع مختلفة من الدوال المغلقة كليا ودرسنا العلاقة فيما بينها. قدمنا تعريفا لهذه الأنواع من الدوال بالإضافة إلى دراسة عدة خواص لهذا الدوال و برهنة العديد من النظريات.

## 1. Introduction

N. Levine [8] in 1963 introduced the concept of semi-open sets, he defined a subset $A$ of a topological space $X$ to be semi-open if there exists an open set $U$ in $X$ such that $U \subseteq A \subseteq \overline{U}$. The complement of a semi-open set is semi-closed. He also introduced the concept of semi-continuous functions. Biswas [2] defined and studied the notion of semi-open functions. Irresolute functions and semi-homeomorphisms were introduced and investigated by Crossly and Hildebrand [5]. Nour [11] defined totally semi-continuous functions. Benchalli and Neeli in [1] introduced and studied semi-totally continuous and semi-totally open functions. Pre semi-closed mapping concept was introduced by Garg and Shivaraj in [7].

The purpose of this paper is to introduce and investigate new types of closed functions; we define totally closed functions and define weak and strong forms of this type of functions. We prove some results in this connection.

## 2. Preliminaries

**Definition 1.** Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is said to be:

1) semi-continuous [8] if the inverse image of each open subset of $Y$ is semi-open in $X$.

2) semi-open [2] if $f(U)$ is semi- open in $Y$ for each open set $U$ in $X$.

3) semi-closed [3] if $f(U)$ is semi- closed in $Y$ for each closed set $U$ in $X$.

4) totally semi-continuous [11] if the inverse image of every open subset of $Y$ is semi- clopen in $X$ .

5) pre semi-open [5] if the image of every semi-open set in $X$ is semi-open in $Y$.

6) pre semi-closed [5] if the image of every semi-closed set in $X$ is semi-closed in $Y$.

7) semi-totally open [1] if the image of every semi-open set in $X$ is clopen in $Y$.

8) irresolute [5] if the inverse image of every semi- open set in $Y$ is semi-open in $X$.

**Definition 2.** [5] The semi-closure of a set $A$ in $X$ is the intersection of all semi-closed sets that contains $A$; this set is denoted by $\underline{A}$.

**Definition 3.** [5] The semi-interior of a set $A$ in $X$ is the union of all semi-open sets of $X$ contained in $A$; this set is denoted by $A_{\circ}$.

**Definition 4.** A topological space $X$ is said to be

1) semi-$T_0$[9] if for each pair of distinct points in $X$, there exists a semi-open set containing one point but not the other.

2) semi-$T_1$ [9] if for each pair of distinct points $x$ and $y$ of $X$, there exist semi-open sets $U$ and $V$ such that $x \in U, y \notin U$ and $x \notin V , y \in V$.

3) semi-$T_2$ [9] if for each pair of distinct points $x$ and $y$ of $X$, there exist semi-open sets $U$ and $V$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

4) $s$-normal [10] if each pair of non-empty disjoint closed sets can be separated by disjoint semi-open sets.

5) $s$-regular [9] if for each closed set $F$ of $X$ and each $x \notin F$, there exist disjoint semi-open sets $U$ and $V$ such that $F \subseteq U$ and $x \in V$.

## 3. Main Results

In this section, new types of totally closed functions are introduced. Also, some relations between these types of functions are obtained. Several results and properties are investigated and proven.

**Definition 5.** A function $f : X \to Y$ is said to be totally closed if the image of every closed set in $X$ is clopen in $Y$.

**Definition 6.** A function $f : X \to Y$ is said to be totally semi-closed if the image of every closed set in $X$ is semi-clopen in $Y$.

**Definition 7.** A function $f : X \to Y$ is said to be semi-totally closed if the image of every semi-closed set in $X$ is clopen in $Y$.

**Definition 8.** A function $f : X \to Y$ is said to be totally pre semi-closed if the image of every semi-closed set in $X$ is semi-clopen in $Y$.

**Theorem 1.** (1) Every semi-totally closed function is totally closed and every totally closed function is totally semi-closed.

(2) Every semi-totally closed function is totally pre semi-closed and every totally pre semi-closed function is totally semi-closed.

*Proof.* The proof of (1) and (2) follows directly from the fact that every clopen set is also semi-clopen and every closed set is also semi-closed.

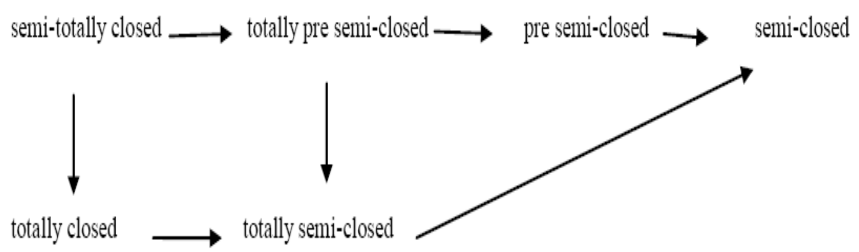
**Observation 1.** The converse of Theorem 1 is not true as shown by the following examples.

**Example 1.** Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}, \{3\}\{1,3\}\}$. Let $Y = \{1,2,3,4\}$ and $\tau^* = \{\phi, Y, \{1\}, \{1,3\}, \{3\}\}$. The function $f: (X, \tau) \to (Y, \tau^*)$ defined by $f(1) = 2, f(2) = 1, f(3) = 4$ is totally semi-closed but not totally pre semi-closed nor totally closed.

**Example 2.** Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{2,3\}\}$ and $\tau^* = \{\phi, X, \{1\}, \{1,3\}, \{3\}\}$. Let $f: (X, \tau) \to (X, \tau^*)$ be the function defined by $f(x) = x. f$ is a totally pre semi-closed function, but $f$ is not semi-totally closed because $\{1\}$ is a semi-closed set in $X$ and $f(\{1\})$ is not clopen in $X$.

**Example 3.** Let $X = \{1,2,3,4\}$ and $\tau = \{\phi, X, \{1\}, \{1,3\}\}$. Let $Y = \{1,2,3\}$ and $\tau^* = \{\phi, Y, \{1\}, \{1,3\}, \{3\}, \{1,2\}\}$. The function $f: (X, \tau) \to (Y, \tau^*)$ defined by $f(1) = f(2) = 1,\ f(3) = 3, f(4) = 2$ is totally closed but not semi- totally closed.

The following diagram illustrates the relation between the different forms of totally closed functions and also other types of closed functions.




**Theorem 2.** (1) The composition of two totally closed functions is totally closed.

(2) The composition of two semi-totally closed functions is semi-totally closed.

(3) The composition of two totally pre semi-closed functions is totally pre semi-closed.

*Proof.* (1) Let $f : X \to Y$ and $g : Y \to Z$ be totally closed functions. Let $F$ be closed in $X$, then $f(F)$ is clopen (and hence is closed) in $Y$ since $f$ is totally closed. Hence, $g(f(U)) = (g \circ f)(U)$ is clopen in $Z$ since $g$ is totally closed. The proof of (2) and (3) follows in the same way.

**Observation 2**. The composition of two totally semi-closed functions do not need to be totally semi-closed as shown by:

**Example 4.** Let $X = \{a, b, c\}, \tau = \{\phi, X, \{b, c\}\}$ and $\tau' = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $Y = \{a, b, c, d\}$ and $\tau^* = \{\phi, Y, \{a\}, \{a, c\}, \{c\}\}$. The function $g: (X, \tau) \to (X, \tau')$ defined by $g(x) = x$ is totally semi-closed. Also, the function $f: (X, \tau') \to (Y, \tau^*)$ defined by $f(a) = b, f(b) = a, f(c) = d$ is totally semi-closed. But the composition $f \circ g: (X, \tau) \to (Y, \tau^*)$ is not a totally semi-closed function because $(f \circ g)(\{a\}) = \{b\}$ and $\{b\}$ is not semi-clopen in $Y$ although $\{a\}$ is closed in $X$.

Similarly to Theorem 2, we can prove the following.

**Theorem 3**. Let $f : X \to Y$ and $g : Y \to Z$ be functions then:

(1) if $f$ is totally closed and $g$ is semi-totally closed (resp. totally semi-closed, totally pre semi-closed) then $g \circ f$ is totally closed ( resp. totally semi-closed, totally semi-closed).

(2) if $f$ is semi-totally closed and $g$ is totally closed (resp. totally semi-closed, totally pre semi-closed) then $g \circ f$ is semi-totally closed (resp. totally pre semi-closed, totally pre semi-closed).

(3) if $f$ is totally semi-closed and $g$ is semi-totally closed (resp. totally pre semi-closed) then $g \circ f$ is totally closed (resp. totally semi-closed).

(4) if $f$ is totally pre semi-closed and $g$ is semi-totally closed then $g \circ f$ is semi-totally closed.

The proof of the following theorem is obvious and hence omitted.

**Theorem 4.** If $f : X \to Y$ is totally closed (resp. totally semi-closed) function and $A$ is a closed subset of $X$, then the restriction function $f|A : A \to Y$ is also totally closed (resp. totally semi-closed).

**Theorem 5.** A function $f : X \to Y$ is totally closed (resp. totally pre semi-closed) if and only if for all $A \subseteq X$, $\overline{f(A)} \subseteq f(\bar{A})$ and $f(A) \subseteq (f(\bar{A}))^{\circ}$ (resp. $\underline{f(A)} \subseteq f(\underline{A})$ and $f(A) \subseteq (f(\underline{A}))_{\circ}$).

*Proof.* Let $f : X \to Y$ be a totally closed function and $A \subseteq X$. Since $A \subseteq \bar{A}$, then $f(A) \subseteq f(\bar{A})$. Since $f(\bar{A})$ is clopen in $Y$ then we have $\overline{f(A)} \subseteq \overline{f(\bar{A})} = f(\bar{A})$ and $f(A) \subseteq f(\bar{A}) = (f(\bar{A}))^{\circ}$.

Conversely, suppose that for all $A \subseteq X$, $\overline{f(A)} \subseteq f(\bar{A})$ and $f(A) \subseteq (f(\bar{A}))^{\circ}$. Let $B$ be a closed subset of $X$, then $\overline{f(B)} \subseteq f(\bar{B}) = f(B)$, so $f(B)$ is closed in $Y$. Also, $f(B) \subseteq \left(f(\bar{B})\right)^{\circ} = (f(B))^{\circ}$, so $f(B)$ is also open. Therefore $B$ is clopen in $Y$.

**Definition 9.** A function $f : X \to Y$ is said to be totally open (resp. totally semi-open, totally pre semi-open) if the image of every open (resp. open, semi-open) set in $X$ is clopen (resp. semi-clopen, semi-clopen) in $Y$.

The proof of the following theorem is obvious.

**Theorem 6.** A bijection $f: X \to Y$ is totally closed (resp. totally semi-closed, semi-totally closed, totally pre semi-closed) if and only if it is totally open (resp. totally semi-open, semi-totally open, totally pre semi-open).

**Theorem 7.** If $f : X \to Y$ is a totally closed (resp. totally semi-closed, semi-totally closed, totally pre semi-closed) bijection and $X$ is a $T_2$ space (resp. $T_2$ space, semi-$T_2$ space, semi-$T_2$space) then $Y$ is a $T_2$ space (resp. semi- $T_2$ space, $T_2$space, semi-$T_2$space).

*Proof.* Let $f : X \to Y$ be a totally closed bijection and let $x, y \in Y$ be distinct points, so $f^{-1}(x) \neq f^{-1}(y)$. Since $X$ is a $T_2$-space, there exist disjoint open sets $U$ and $V$ such that $f^{-1}(x) \in U$ and $f^{-1}(y) \in V$. Since $f$ is a totally closed bijection and From Theorem 6, $f(U), f(V)$ are disjoint open sets and $x \in f(U)$, $y \in f(V)$.

*Corollary 1.* If $f : X \to Y$ is a totally closed (resp. totally semi-closed, semi-totally closed, totally pre semi-closed) bijection and $X$ is a $T_i$ space (resp. $T_i$ space, semi-$T_i$space, semi-$T_i$space), $i = 0,1$ , then $Y$ is a $T_i$ space (resp. semi- $T_i$ space, $T_i$ space, semi-$T_i$space).

**Theorem 8**. Let $f : X \to Y$ be a continuous injection, if $f$ is totally closed or semi-totally closed then $X$ is a normal space.

*Proof*. Let $F_1$ and $F_2$ be disjoint closed sets in $X$. If $f$ is a totally closed (or semi-totally closed) injection, then $f(F_1)$, $f(F_2)$ are disjoint clopen sets in $Y$ and $F_1 = f^{-1}(f(F_1))$, $F_2 = f^{-1}(f(F_2))$. Since $f$ is continuous, $F_1$ and $F_2$ are open sets, hence the proof is complete.

Similarly, we can prove the following theorem.

**Theorem 9**. Let $f : X \to Y$ be an irresolute injection, if $f$ is totally semi-closed or totally pre semi-closed then $X$ is an $s$-normal space.

**Theorem 10**. Let $X$ be a $T_1$ space and $f : X \to Y$ be a continuous injection. If $f$ is totally closed or semi-totally closed then $X$ is also regular.

*Proof*. Let $F$ be a closed set in $X$ and $x \notin F$. Since $X$ is a $T_1$ space, $\{x\}$ is closed in $X$. The rest of the proof is similar to that of Theorem 8.

**Theorem 11**. Let $X$ be a $T_1$ space and $f : X \to Y$ be an irresolute injection. If $f$ is totally semi-closed or totally pre semi-closed then $X$ is also $s$-regular.

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