T_ψ Spaces and ψ- Normality

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تاريخ الاستلام 2023/10/25

الملخص
تم في هذا البحث تطبيق مجموعات ψ المغلقة لتعريف فئة جديدة من الفضاءات والتي تم تسميتها فضاءات T_ψ وتقع بين فضاءات T_b وفضاءات semi-$T_{1/3}$، كما تم دراسة بعض خصائصها وعلاقتها بالفضاءات T_b، $T_{1/2}$* و T_c . علاوة على ذلك، يتم هنا تقديم ودراسة نوع جديد من الناظمية تُسمى ψ-normal spaces.

Abstract

Applying ψ-closed sets a new class of spaces namely, T_ψ spaces is introduced which is properly placed in between T_b and semi-$T_{1/3}$, and studying the relationships between T_b, $T_{1/2}$*, T_c spaces and T_ψ spaces. Moreover, a new type of normality is introduced and studied here, that is the ψ-normal spaces.

Keywords: ψ-closed set, quasi ψ-closed map, $T_{1/2}$* spaces, T_ψ spaces, ψ-normal spaces.

Introduction

N. Levine (Levine N. , 1963) introduced semi-open sets, and in (Levine N. , 1970) he generalized the concept of closed sets to generalized closed sets. Bhattacharya and Lahiri (Bhattacharya & Lahiri, 1987) generalized the concept of closed sets to semi-generalized closed sets. S.P.Arya and T.Nour (Arya & Nour, 1990) defined gs-closed sets in 1990. M.V. Kumar (Kumar, M.K.R.S. Veera, 2000) introduced the ψ-closed sets and he defined the semi-$T_{1/3}$ space as an application of ψ-closed sets. In (Tawfik, 2007) the classes of quasi ψ-closed maps, and strongly ψ-closed maps have been defined by using ψ-closed sets due to Kumar. In 1993 Devi et.al (Devi, Maki, & Balachandran, 1993) defined T_b spaces. M.V. Kumar (Kumar, M.K.R.S. Veera, 2000) introduced g*-closed sets and new classes of maps namely g*-continuous maps, g*-irresolute maps and pre- g*-closed maps. Applying g*-closed sets, in (Kumar, M.K.R.S. Veera, 2000) four new spaces namely, $T_{1/2}$* spaces, *$T_{1/2}$ spaces, T_c spaces and $_\alpha T_c$ spaces are introduced. Applying ψ-closed sets a new class of spaces namely, T_ψ spaces is introduced here, which is properly placed in between T_b and semi-$T_{1/3}$, and studying the relationships between T_b, $T_{1/2}$*, T_c spaces and T_ψ spaces. Moreover, a new type of normality is introduced and studied here, that is the ψ-normal spaces.

1. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, cl(A) and int(A) denote the closure and the interior of A respectively.

Definition 2.1: –

A subset A of a topological space X is called

1) a *semi-open* set (Levine N. , 1963) if $A \subseteq$ cl(int(A)) and a *semi-closed* set if int (cl(A)) $\subseteq A$.

Definition 2.3: –

A subset A of a topological space X is called

1) a *generalized closed* set (briefly *g-closed*) (Levine N. , 1970) if cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is open in X.
2) a *semi-generalized closed* set (briefly *sg-closed*) (Bhattacharya & Lahiri, 1987) if scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
3) a *generalized semi-closed* set (briefly *gs-closed*) (Arya & Nour, 1990)if scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is open in X.
4) a *g*-closed* set (Kumar, M.K.R.S. Veera, 2000) if cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is g-open in X.
5) a ψ *-closed* set (Kumar, M.K.R.S. Veera, 2000) if scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is a sg-open set of X.

The class of all closed (resp. semi-closed) subsets of a space X is denoted by C(X) (resp. SC(X)). The intersection of all semi-closed sets containing a subset A of X is called the semi-closure of A and is denoted by scl(A). The class of all g-closed (resp. gs-closed, sg-closed, g*-closed) sets of a space X is denoted by GC(X) (resp. GSC(X), SGC(X), G*C(X)). The complement of g-closed (resp. gs-closed, sg-closed, g*-closed, ψ-closed) sets is g-open (resp. gs- open, sg- open, g*- open, ψ-open) sets. The class of all ψ -closed sets of a space X is denoted by ΨC(X). The intersection of all ψ -closed sets containing a subset A of a space X is called a ψ -closure and is denoted by ψcl(A). The class of ψ -closed sets properly contains the class of semi-closed sets, and thus contains the class of closed sets. Also, the class of ψ -closed sets is properly contained in the class of sg-closed sets, and contained in the class of gs-closed sets. Theorem 3.3 of (Kumar, M.K.R.S. Veera, 2000).

Definition 2.2: –

A function $f: X \to Y$ is said to be

1) *continuous* (Joshi, 1983) if $f^{-1}(V)$ is open set of X for every open set V of Y.
2) *semi-continuous* (Levine N. , 1963) if $f^{-1}(V)$is a semi-open set of X for every open set V of Y.
3) ψ-continuous (Kumar, M.K.R.S. Veera, 2000) if $f^{-1}(V)$ is a ψ -closed set of X for every closed set V of Y.
4) *g*-continuous* (Kumar, M.K.R.S. Veera, 2000) if $f^{-1}(V)$ is a g*-closed set of X for every closed set V of Y.
5) *ψ-irresolute* (Kumar, M.K.R.S. Veera, 2000) if $f^{-1}(V)$ is a ψ -closed set of X for every ψ-closed set V of Y.
6) *g*-irresolute* (Kumar, M.K.R.S. Veera, 2000) if $f^{-1}(V)$ is a g*-closed set of X for every g*-closed set V of Y.
7) *closed* (Joshi, 1983) if $f(V)$ is closed set of Y for every closed set V of X.
8) *quasi ψ-closed* (Tawfik, 2007) if $f(V)$ is closed set of Y for every ψ-closed set V of X.
9) *ψ-closed* (Tawfik, 2007) if $f(V)$ is ψ-closed set of Y for every closed set V of X.
10) *strongly ψ-closed* (Tawfik, 2007) if $f(V)$ is ψ-closed set of Y for every ψ-closed set V of X.
11) *pre-g*-closed* (Kumar, M.K.R.S. Veera, 2000) if $f(V)$ is a g*-closed set of Y for every g*-closed set V of X.

Definition 2.3: –

A topological space X is said to be

(1) a $T_{1/2}$ space (Levine N. , 1970) if every g-closed set in it is closed.

(2) a *semi-*$T_{1/2}$ space (Bhattacharya & Lahiri, 1987) if every sg-closed set in it

is semi-closed.

(3) a *semi*-$T_{1/3}$ space (Kumar, M.K.R.S. Veera, 2000) if every ψ -closed set in it is semi-closed.

(4) a T_b space (Devi, Maki, & Balachandran, 1993) if every gs-closed set in it is closed.

(5) a T_d space (Devi, Maki, & Balachandran, 1993) if every gs-closed set in it is g-closed.

(6) $T_{1/2}$* space (Kumar, M.K.R.S. Veera, 2000) if every g*-closed set in it is closed.

(7) T_c space (Kumar, M.K.R.S. Veera, 2000) if every gs-closed set in it is g*-closed.

Theorem 2.1: - (Kumar, M.K.R.S. Veera, 2000)

Every T_b space is a $T_{1/2}$* space.

Theorem 2.2: - (Kumar, M.K.R.S. Veera, 2000)

Every T_b space is a T_c space but not conversely.

Theorem 2.3: - (Kumar, M.K.R.S. Veera, 2000)

A space X is a T_b space if and only if it is a T_c and a $T_{1/2}$* space.

Theorem 2.4: -

Every T_b space is a semi-$T_{1/3}$ space.

Proof:

Let A be a ψ-closed set of space X, hence it is a gs-closed set (Kumar, M.K.R.S. Veera, 2000) then it is closed since X is a T_b space. So, A is a semi-closed set since every closed set is semi-closed.

Theorem 2.5: - (Kumar, M.K.R.S. Veera, 2000)

Every semi-$T_{1/2}$ space is a semi-$T_{1/3}$ space.

T_ψ Spaces

By applying ψ-closed set we define a new class of spaces

Definition 3.1: –

A topological space X is said to be T_ψ space if every ψ -closed set in it is closed.

Theorem 3.1: -

Every T_b space is a T_ψ space.

Proof:

Let A be a ψ-closed set in space X, then it is a gs-closed set, hence it is closed since X is a T_b space.

The following example shows that the converse of the above theorem is not true in general.

Example 3.1: -

Let $X = \{a, b, c\}$, $\tau = \{\phi , X, \{a\}, \{b, c\}\}$. X is T_ψ space but not a T_b space since {a,b} is gs-closed but not closed.

By theorem 3.1 and theorem 2.4 we get the following

Theorem 3.2: -

Every T_c and a $T_{1/2}{}^*$ space is T_ψ space.

Remark:

T_ψ ness is independent from T_c ness as it can be seen from the following examples.

Example 3.2: -

Let X as in the example 2.1. X is not T_c space since {a,b} is gs-closed but not g*-closed, whenever it is T_ψ space.

Example 3.3: -

Let $X = \{a, b, c,d\}$, $\tau = \{\phi , X, \{b,c\},\{a,b,c\},\{b,c,d\}\}$. X is T_c space but not a T_ψ space since {a,b,d} is ψ-closed set but not closed set.

Theorem 3.3: -

Every T_ψ space is a semi-$T_{1/3}$ space.

Proof:

Let A be a ψ-closed set in a T_ψ space X, then it is a closed set hence is semi-closed.

The following example shows that the converse of the above theorem is not true in general.

Example 3.4: -

Let $X = \{a, b, c\}$, $\tau = \{\phi , X, \{a\}, \{a, c\}\}$. X is semi-$T_{1/3}$ space but it is not a T_ψ space since {c} is ψ-closed set but it is not a closed set.

Remark:

T_ψ ness is independent from semi-$T_{1/2}$ ness as it can be seen from the following examples.

Example 3.5: -

Let X as in the example 2.1. X is not semi-$T_{1/2}$ space since {b} is sg-closed but not semi-closed, whenever it is T_ψ space.

Example 3.6: -

Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$. X is semi-$T_{1/2}$ space but not a T_ψ space since {b} is ψ-closed set but not closed set.

Theorem 3.4: -

Every T_d and $T_{1/2}$ space is a T_ψ space.

Proof:

Since every ψ-closed set is gs-closed (Kumar, M.K.R.S. Veera, 2000), hence is g-closed set because of T_d property, and hence is closed because of $T_{1/2}$ property. Therefore T_ψ property is satisfied.

By virtue of theorem 3.3 and [theorem 4.4 (Kumar, M.K.R.S. Veera, 2000)] we can characterize the T_ψ spaces as follows

Theorem 3.5: -

For the topological space X the following conditions are equivalent

1) X is T_ψ space.
2) Every singleton of X is either sg-closed or semi-open.
3) Every singleton of X is either sg-closed or open.

ψ-normal Spaces

Definition 4.1: –

A topological space X is said to be ψ-normal if for every two disjoint closed sets A, B there exist two disjoint ψ-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.1: -

Let X be ψ-normal. Then for each closed set A and each open set B containing A there exists a ψ-open set G such that $A \subseteq G \subseteq scl\ G \subseteq B$.

Proof:

Let $A \subseteq X$ be a closed set and $B \subseteq X$ be an open set such that $A \subseteq B$. Since X is ψ-normal then there exist two disjoint ψ-open sets G and H such that $A \subseteq G$

and $X\backslash B \subseteq H$, i.e. $X\backslash H \subseteq B$ and since G and H are disjoint then $G \subseteq X\backslash H$ which is ψ-closed and so, we have $A \subseteq G \subseteq X\backslash H \subseteq \psi cl(G) \subseteq scl(G) \subseteq B$. This completes the proof.

Theorem 4.2: -

Let *X*, *Y* two topological spaces. If *X* is ψ-normal space and $f: X \longrightarrow Y$ is continuous strongly ψ-closed bijection, then *Y* is ψ-normal also.

Proof:

Let *A* and *B* are two disjoint closed sets of *Y* then $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint closed sets of *X*, then there exist two disjoint ψ-open sets *U* and *V* such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$, so that $A \subseteq f(U)$ and $B \subseteq f(V)$ where $f(U)$ and $f(V)$ are disjoint ψ-open sets of *Y*.

Theorem 4.3: -

If $f: X \longrightarrow Y$ is closed ψ-irresolute injection of a topological space *X* to a ψ-normal space *Y*, then *X* is ψ-normal.

Proof:

Let *A* and *B* are two disjoint closed sets of *X* then $f(A)$ and $f(B)$ are two disjoint closed sets of *Y*. Since *Y* is ψ-normal then there exist two disjoint ψ-open sets *U* and *V* such that $f(A) \subseteq U$ and $f(B) \subseteq V$, so that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$ where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint ψ-open sets of *X*.

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