HARMONIC FUNCTION

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الملخص

في هذا البحث قدمنا فكرة عن الدوال التوافقية، لذلك قدمنا أولا أهم النظريات المتعلقة بالدوال التوافقية، وطرح فكرة المرافق التوافقي. وقدمنا أيضا مفهوم جرين ومسألة ديريشليه. وخلصنا إلى أن الدوال التوافقية لها العديد من التطبيقات يمكن استخدامها في دراسة القرص وفهم مسألة ديريشليه.

الكلمات المفتاحية: الدوال التوافقية، المرافق التوافقي، مسألة ديرشليه.

Abstract

In this paper, we present an innovative idea of the harmonic functions. In order to do this, we first present the most important theories related to harmonic functions and put forward the idea of harmonic conjugate. We also presented the concept of Green's identity and Dirichlet problem. We concluded that harmonic functions have many applications and can be used in studying the disk and understanding the Dirichlet problem.

Keywords: harmonic functions, harmonic conjugate, Dirichlet problem.

1. Introduction

Harmonic functions are important in the areas of Mathematical physics, applied mathematics, engineering, electricity and magnetism. Harmonic

functions are used to solve problems involving fluid flow, constant temperature, two-dimensional electricity, and to improve imaging accuracy.

Over the past ages, several ways discovered to prove a solution to the Dirichlet problem of the Laplace equation. The set of citation proofs is based on representations of the Green's function in terms of a Bergman kernel function or some equivalent linear parameter.

We will study how complex analysis can be used to solve problems related to harmonic functions.

Definition

If H is a real function of two real variables, x and y are compatible in a given domain of the xy level, H along that range contains continuous first and second order partial derivatives, and satisfies the partial differential equation

$$\nabla^2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

Known as Laplace's equation.

Remark

(i) Harmonic functions often arise when we study the issue of electrostatic fields as well as gravity and thermal conductivity in the steady state, incompressible fluid flows etc..

(ii) Harmonic functions are technically similar to holomorphic functions. If we take, for example, the function is a holomorphic function, let $f = u + \imath v$. It is a result of Cauchy-Riemann equations. Therefore, we find that both u and v are harmonic. We call v harmonic conjugates of u, and the $f\,(x, y)$ function verifies that u is a harmonic conjugate to $-v$.

(iii) For the definition of harmonic functions, there is nothing very special about considering only real valued functions. As in the above definition, we could even allow complex valued functions. The complex function is harmonic if its real part and imaginary part are harmonic. Thus, it is

enough to treat only the real valued functions, in the study of harmonic functions.

(iv) From the linearity of the differential operator ∇^2, it follows that the set of all harmonic functions on a domain forms a vector space. In particular all linear functions $ax + by$ are harmonic. However, it is not true that product of two harmonic functions is harmonic. For example, xy is harmonic but x^2y^2 is not.

(v) Harmonicity is quite a delicate property. If φ is a smooth real valued function of a real variable and u is harmonic, then, in general, $\varphi\ o\ u$ need not be harmonic. Indeed, $\varphi\ o\ u$ is harmonic for all harmonic u if φ is linear. Likewise, if $f : \Omega_1 \to \Omega_2$ is a smooth complex valued function of two real variables then $u\ o\ f$ need not be harmonic.

Example

Consider the function $u: R^2 \to R$ given by

$$u(x, y) = x^3 - 3xy^2 - 2y$$

Show that u is harmonic on R^2.

Solution

We calculate the following partial derivatives of u:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 , \qquad \frac{\partial u}{\partial y} = -6xy - 2$$

And

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

Observe that the second-order partial derivatives are continuous and satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is harmonic on R.

Theorem

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

To prove this, we need a corollary in the following:

Corollary

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = (x, y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.

Let f is analytic in D. We apply the Cauchy–Riemann condition by finding the first-order partial derivatives of the function in the all points of the domain D.

$$u_x = v_y, \quad u_y = -v_x \qquad (2)$$

Differentiating both, sides of these equations with respect to x, we have

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx} \qquad (3)$$

Likewise, differentiation with respect to y yields

$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy} \qquad (4)$$

Now, that the partial derivatives of u and v are continuous, by theorem in advanced calculus ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. It then follows from equations (3) and (4) that

$$u_{xx} + u_{yy} = 0 \quad \text{And} \quad v_{xx} + v_{yy} = 0$$

That is, u and v are harmonic in D.

2. Harmonic Conjugate

Definition

Let u and v are harmonic functions that satisfy the Cauchy-Riemann equations on some open set. In other words, the function $f = u + iv$ is analytic in open set. We call v the harmonic conjugate of u.

This definition will lead us to ask the following question: Can we always find the harmonic conjugate of the harmonic function u?

The answer of this question depends on the function u and its domain of definition. For example, the function $ln|z|$ is harmonic in $\Omega = C\backslash\{0\}$ but $ln|z|$ has no harmonic conjugate in that region. However, have a harmonic conjugate in $C\backslash(-\infty, 0]$, namely Argz.

Proposition

Suppose that u is harmonic on an open set Ω, and that v is a harmonic conjugate of u on Ω. Then $-u$ is a harmonic conjugate of v on Ω.

Proof

We know that $f = u + iv$ is analytic. It follows that the function $(-i)f = v - iu$ is analytic. Hence $-u$ is a harmonic conjugate of v on Ω.

Proposition

Suppose that u is a harmonic function in a region Ω. Then

(i) If v_1 and v_2 are harmonic conjugates of u in Ω, Therefore, we find that v_1 differs from v_2 by means of a real constant.

(ii) If v is a harmonic conjugate of u then v is also a harmonic conjugate of $u + c$ where c is a constant.

3. Green's identity

Green's Identity is a very important tool in solving harmonic functions and Laplace's equation. This identity is derived from the divergence theorem, so we will first present the divergence theorem.

Theorem Divergence theorem in R^n-Greens

Suppose that Ω be a bounded domain with C^1 boundary $\partial\Omega$ in R^n. Then for any vector field $\omega \in C^1(\bar{\Omega})$ we have

$$\int_\Omega div(\omega)dV = \int_{\partial\Omega} \omega . v ds$$

Where ds is the area element in $\partial\Omega$ and ν the unit outward normal vector to $\partial\Omega$.

Theorem Green's identity

Suppose that Ω be a bounded subset of R^n with smooth boundary $\partial\Omega$. Suppose that u and v are C^2- functions on a neighborhood of $\overline{\Omega}$. Then Ω , be a bounded subset of R^n with smooth boundary $\partial\Omega$. Let u and v are C^2-functions on a neighborhood of $\overline{\Omega}$. Then

$$\int_{\Omega}(u\Delta v - v\Delta u)dV = \int_{\partial\Omega}(uD_\nu v - vD_\nu u)ds$$

Theorem

Suppose that γ to be a simple closed path with positive direction. Let D is the inside region of γ.take $P(x,y)$ and $Q(x,y)$ with their first-order partial derivatives side by side are real and continuous functions of $\overline{D} = D \cup \gamma$. then

$$\int_{\gamma} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

We note that the integral on the right side is a double integral over area D, so that the left side is simplified and more understood. Parameterize γ by $x = x(t)$ and $y = y(t)$, where $a \le t \le b$. Then we integrate the left side, we get

$$\int_a^b \Big(P\big(x(t), y(t)\big)x'(t) + Q\big(x(t), y(t)\big)y'(t)\Big)\, dt.$$

4. Dirichlet problem

Suppose that Ω be a bounded domain in $R^n (n \ge 2)$, let u be a continuous real-valued function on its boundary $\partial\Omega$. The classical Dirichlet problem consists in the determination of a harmonic function u on Ω which can be continuously extended into $\partial\Omega$ by u. We note that the Dirichlet problem is a problem of the Laplace equation on the boundary.

Let v is a given continuous function on the boundary $\partial\Omega$ of a domain $\Omega \subset R^n$. The Dirichlet problem is to extend v (which is only defined on the

boundary $\partial\Omega$ of Ω) to a function u defined inside the domain Ω, such that (i) $\Delta u = 0$ in Ω (i.e. u is harmonic).

(ii) $u = v$ On $\partial\Omega$ (more precisely for all $y \in \partial\Omega$ we want $u(x) \to v(y)$ as $x \to y$ where $x \in \Omega$).

The Poisson kernel is very important in the Dirichlet problem.

The solution to a Dirichlet problem is always in spheres with homogeneous boundaries, for example a disk or a sphere.

We will now present the following Dirichlet problem:

Suppose that Ω be an open and bounded subset of R^n

$$\begin{cases} -\Delta u = f & x \in \Omega \subset R^n \\ u = g & x \in \partial\Omega \end{cases} \qquad (6)$$

We will first introduce Green's function because we will use it to solve the Dirichlet problem.

Consider the following problem:

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases} \qquad (7)$$

The first aim is to get as G so we have

$$u(x) = \int_\Omega \delta_x u(y)dy$$

$$= -\int_\Omega \Delta_y G(x,y)u(y)dy$$

$$= -\int_\Omega \langle \nabla_y G(x,y), \nabla_y u(y)\rangle dy - \int_\Omega \frac{\partial G}{\partial v}(x,y)u(y)dS(y)$$

$$= -\int_\Omega G(x,y), \Delta_y u(y)dy + \int_{\partial\Omega} G(x,y)\frac{\partial u}{\partial v}(y)dS(y) - \int_\Omega \frac{\partial G}{\partial v}(x,y)u(y)dS(y)$$

$$= \int_\Omega G(x,y)f(y)dy - \int_\Omega \frac{\partial G}{\partial v}(x,y)g(y)dS(y)$$

Where we have used $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$, and $G(x,y) = 0$ on $\partial\Omega$.

Here ν is the outer unit normal to Ω.
We know that

$$-\Delta_y \varphi(y) = \delta_0$$

$$-\Delta_y \varphi(x-y) = \delta_x$$

Where, φ is a solution of the Laplace equation.

$$\varphi(x) = \begin{cases} -\frac{1}{2\pi} ln|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)|x|^{n-2}} & n \geq 3 \end{cases}$$

We note that $\varphi(x - y)$ does not satisfy the given boundary conditions, but it can still be used to help find a solution integrating.

Proposition

Let u and v be harmonic on an open set S, and a, b are real constants. We note that the function $au + bv$ is harmonic on S.

Proposition Dirichlet Problem on a Disk

The solution of the Dirichlet problem on the disk $|z| \leq R$ with boundary condition

$$f(\theta) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\theta) + b_n \sin(n\theta)) \quad (8)$$

Is

$$u(re^{i\theta}) = a_0 + \sum_{n=1}^{N} \left(\frac{r}{R}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)), \quad r < R \quad (9)$$

Proof

For $|z| < R$, write $z = re^{i\theta} = r(cos\theta + isin\theta)$, the function

$$f(z) = z^n = r^n(\cos(n\theta) + isin(n\theta))$$

Is analytic on the disk $|z| < R$.

We conclude from this that the real part is harmonic as well as the imaginary part is also harmonic, and this leads us to

$1, r\cos\theta, r^2\cos(2\theta), \ldots, r\sin\theta, r^2\sin(2\theta), \ldots$ are harmonic in the variable $re^{i\theta}$ on the disk $B_R(0)$.

We know that if u and v are harmonic on an open set S, and a, b are real constants. Then the function $au + bv$ is harmonic on S. it leads us to the fact that the linear combinations of these functions are also harmonic on the unit disk, and so (9) is harmonic. Setting $r = R$ in (9), we see that $u(Re^{i\theta}) = f(\theta)$ where f is as in (8). Hence (9) is the solution of the Dirichlet problem with boundary data (8)

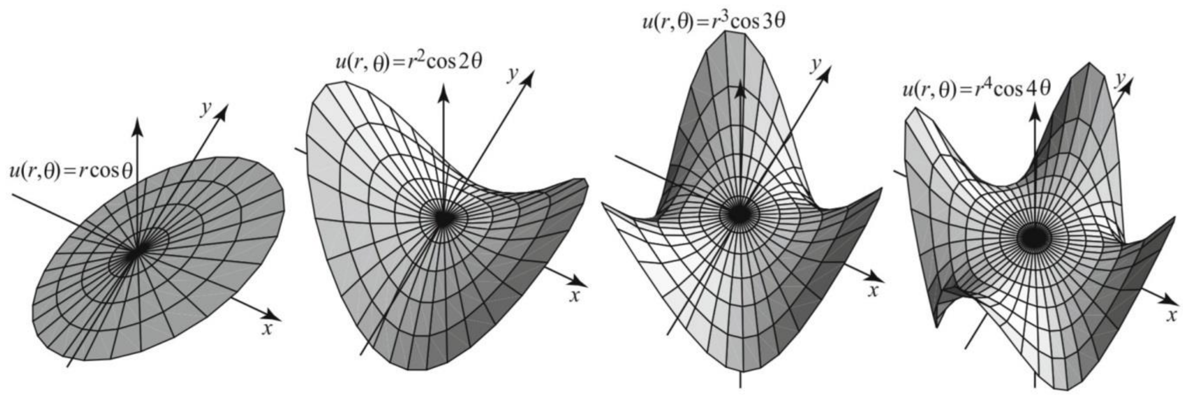

Fig 1 The saddle-shaped graphs of the harmonic functions $r\cos\theta$, $r^2\cos 2\theta$, $r^3\cos 3\theta$, $r^4\cos 4\theta$, respectively.

Each term in the finite sum (9) is a constant multiple of the harmonic functions $1, r\cos\theta, r^2\cos(2\theta), \ldots, r\sin\theta, r^2\sin(2\theta), \ldots$. an exception to this is the static function1 . Note that the graphs of these functions over the disk $|z| < R$ are saddle- shaped; see Figure 1.

5. Conclusions and Recommendations

We concluded that harmonic functions are of great importance and have many important applications.

We recommend studying the solution of Dirichlet problems using the theorems of the Poisson integral equation and Fourier series, because it gives a development to the theory of harmonic functions and is more general, and we recommend studying the applications of harmonic functions.

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