Studying smoothing effects for a transport diffusion equation

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الملخص

في هذا البحث سوف ندرس التقدير التفاضلي لمنظومة معادلة النقل التوزيعي التي تحتوي على مؤثر تفاضلي كسري وفي فضاء بزوف الدالي $B^0_{2,1}$ (تعريف 2.2.4). سوف نستعمل للبرهان التقريب الذي يعتمد على إحداثيات لاجرانج مقارنة بالتحليل شبه التفاضلي.

Abstract

In this paper, we study a smoothing effect for a transport diffusion equation, with a fractional dissipation, and in the Besov spaces $B^0_{2,1}$ given in Definition 2.2.4. We use a new approach based on Lagrangian coordinates combined with paradifferential calculus.

Keywords: Transport diffusion equation, Incompressible fluid flow, smoothing effects.

1 Introduction

In this research, we are concerned with the initial value problem of the 2D dissipative quasi-geostrophic model

$$\begin{cases} \partial_t\theta \ + v\cdot\nabla\theta \ + (|D|^{\alpha} + I)\theta \ = f \\ div\ v = 0, \\ \theta|_{t=0} = \theta_0, \end{cases} \qquad (1.1)$$

where θ is the scalar function represents the potential temperature and the parameter $\alpha \epsilon [0,1]$. The $2D$ velocity field $v = v(x,t),\ x \in \mathbb{R}^2,\ t \in \mathbb{R}_+$, is determined by Riesz transform $R_i, \forall i = 1,2$ of θ, that is

$$v = \left(-\frac{\partial_2}{|D|}\theta\,, \frac{\partial_1}{|D|}\theta\right) := \ (-R_2\theta\ ,\ R_1\theta).$$

The differential operator $v\cdot\nabla$ is defined respectively by

$$v.\nabla = \sum_{i=1}^{d} v_i\partial_i.$$

The fractional differential operator $|D| = (-\Delta)^{\frac{1}{2}}$ is defined by its Fourier transform

$$\mathcal{F}(|D|u) = |\xi|\mathcal{F}(u),$$

and the operator $div\ v$ is defined by

$$div\ v = \sum_{i=1}^{2} \partial_{\mathrm{i}} v^{\mathrm{i}}\,.$$

The first equation of (1.1) serves as a $2D$ models arising in geophysical fluid dynamic [11] and the second equation $div\, v = 0$, describe the incompressibity of the fluid.

This equation has been intensively investigated and much attention is carried to the problem of global well-posedness. For the sub-critical case ($\alpha > 1$) the theory seems to be in a satisfactory state. Indeed, global existence and uniqueness for arbitrary initial data are established in various function spaces (see for example [6]). However, the critical and super-critical cases, corresponding respectively to $\alpha = 1$ and $\alpha < 1$, are harder to deal with. In the super-critical case $\alpha < 1$, we have until now only global results for small initial data, see for instance [2] and [10]. For critical case $\alpha = 1$, Constantin, C´ordoba and Wu showed in [5] the global existence in Sobolev space H^1 under smallness assumption of L^∞ norm of θ_0 but the uniqueness is proved for initial data in H^2.

The main goal of this work is to establish a smoothing effect for a transport diffusion model in the critical case, that is when the initial data belong to the homogeneous critical Besov space.

The paper is organized as follows. In section 2, we give some definitions and recall some functional spaces. Section 3 is devoted to recall some well-known results, that will be need in the next section. In section 4, results and discussion are shown, and some conclusions are drawn in section 5.

2 Technical Tools

In this section, we recall some notations and some functional spaces as a Lebesgue space L^p, and Besov space and some results used in the paper.

2.1 Notation

- We denote by C any positive constant than will change from line to line and C_0 a real positive constant depending on the size of the initial data.
- For any A and B, we say that $A \lesssim B$, if there exist a constant $C > 0$ such that $A \leq CB$.
- The space C_0^∞ is the space of all continuous function.

2.2 Some functional spaces

This subsection is devoted to recall some functional spaces and some important results. The following definition see [1].

Definition 2.2.1

A continuous map $f: X \to Y$ is homeomorphism, if it is bijective and its inverse is continuous.

Definition 2.2.2 [1] and [3]

We define the flow associate to the velocity v by the following:

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau) d\tau$$

Definition 2.2.3 [3] and [7]

We define the usual Lebesgue space $L^p(\mathbb{R}^d), p \in [1, +\infty[$, by the space of all function f such that:

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

and for $p = \infty$, we say that $f \in L^\infty$, if

$$\|f\|_{L^\infty} := sup_x |f(x)| < \infty.$$

We need the definition of Besov space. We define the dyadic decomposition of the full space $\mathbb{R}^2$ and recall the Littlewood-Paley operators, see for example [1] and [3]. There exist two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2/\{0\})$ such that :

$$\chi(\xi) + \textstyle\sum_{q\geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{q\in\mathbb{Z}} \varphi(\, 2^{-q}\xi) = 1, \forall \xi \in \mathbb{R}^2/\{0\},$$

$$|p-q| \geq 2 \Rightarrow supp\, \varphi(\, 2^{-p}.) \cap\, upp\, \varphi(\, 2^{-q}.) = \phi,$$

$$q \geq 1 \Rightarrow supp\, \chi \cap\, upp\, \varphi(\, 2^{-q}.) = \phi.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\bar{h} = \mathcal{F}^{-1}\chi$, the frequency localization operators Δ_q and S_q are defined by :

$$\Delta_q f = \varphi(2^{-q}D)f, \qquad S_q f = \chi(2^{-q}D)f$$

$$\Delta_{-1} f = S_0 f, \qquad \Delta_q f = 0 \quad for\ q \leq -2.$$

The homogeneous operators are defined as follows

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q f = \varphi(2^{-q}D)f.$$

We recall now the definition of Besov spaces, see [1] and [3].

Definition 2.2.4 (Besov space)

Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}$ defined by:

$$B^s_{p,r} = \left\{ f \in \acute{S}(\mathbb{R}^2) \colon \|f\|_{B^s_{p,r}} < \infty \right\},$$

where S is the Schwartz space and

$$\|f\|_{B^s_{p,r}} \coloneqq \left(2^{qs} \left\| \Delta_q f \right\|_{L^p} \right)_{l^r}.$$

The homogeneous norm :

$$\|f\|_{\dot{B}^s_{p,r}} \coloneqq \left(2^{qs} \left\| \dot{\Delta}_q f \right\|_{L^p} \right)_{l^r}.$$

Definition 2.2.5 [3] and [7]

Let $T > 0$ and $\rho \geq 1$, we denote by $L_T^\rho B_{p,r}^s$ the space of distribution f such that

$$\|f\|_{L_T^\rho B_{p,r}^s} := \left\| (2^{qs} \|\Delta_q f\|_{L^p})_{\ell^r} \right\|_{L_T^\rho} < +\infty.$$

Besides the usual mixed space $L_T^\rho B_{p,r}^s$, we also need Chemin-Lerner space $\tilde{L}_T^\rho B_{p,r}^s$ which defined as the set of all distributions f satisfying

$$\|f\|_{\tilde{L}_T^\rho B_{p,r}^s} := \left\| 2^{qs} \|\Delta_q f\|_{L_T^\rho L^p} \right\|_{\ell^r} < +\infty.$$

Lemma 2.2.1 (Holder inequality) [1] and [3]

If (f, g) belongs to $L^p \times L^q$ for any $(p, q, r) \in [1, \infty]^3$, and such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then fg belongs to L^r and satisfies

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

The following lemma is needed in the proof of our main result see [1] for a proof.

Lemma 2.2.2 (Gronwall's lemma)

Let f is a nonnegative continuous function on $[0, t]$, a is a positive real number and let A be a continuous function on $[0, t]$. Suppose also that:

$$f(t) \leq a + \int_0^t A(\tau) f(\tau) d\tau.$$

Then we have,

$$f(t) \leq a \, exp\left(\int_0^t A(\tau) d\tau \right).$$

3 Some well-known results

In this section, we recall some well-known results, that will be need in the next. Firstly, we give the Bernstein inequalities. This inequality proved in [3] for any tempered distribution u, and the first author S. Sulaiman [13] and [14], proved the same inequality but for the bloc dyadic S_q and $\dot{\Delta}_q$.

Lemma 3.1 (Bernstein lemma)

There exists a constant $C > 0$ such that for all $q \in \mathbb{Z},\ k \in \mathbb{N}$ and for every tempered distribution u we have,

$$sup_{|\alpha|=k} \left\| \partial^\alpha S_q u \right\|_{L^b} \leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \left\| S_q u \right\|_{L^a}, \quad b \geq a \geq 1 \quad (2.1)$$

$$C^{-k} 2^{qk} \left\| \dot{\Delta}_q u \right\|_{L^a} \leq sup_{|\alpha|=k} \left\| \partial^\alpha \dot{\Delta}_q u \right\|_{L^a} \leq C^k 2^{qk} \left\| \dot{\Delta}_q u \right\|_{L^a} \quad (2.2)$$

The following lemma is useful to our result, see [9], for a proof.

Lemma 3.2

Let f be any function in Schwartz class and ψ a diffeomorphism preserving Lebesgue measure, then for all $p \in [1,+\infty]$, and for all $j,q \in \mathbb{Z}$, we have

$$\left\|\dot{\Delta}_j(\dot{\Delta}_q f o\psi)\right\|_{L^p} \leq C2^{|j-q|}\left\|\nabla\psi^{\alpha(j,q)}\right\|_{L^\infty}\left\|\dot{\Delta}_q f\right\|_{L^p},$$

with

$$\alpha(j,q) = sign(j-q).$$

The following result is proved in [3] and [9].

Lemma 3.3

Let u any smooth function in the Besov space $B^0_{2,1}$, and v be a divergence free vector field of $\mathbb{R}^2$. Then we have

$$\sum_{q\in\mathbb{Z}}\left\|[\Delta_q, v.\nabla]u\right\|_{L^2} \lesssim \|\nabla v\|_{L^\infty}\|u\|_{B^0_{2,1}}.$$

The proof of the following result can be found in [9].

Proposition 3.1

If $f \in \dot{B}^\alpha_{2,1}$ such that $\alpha \in [0,1[$, and let ψ be a Lipschitz measure-preserving homeomorphism on $\mathbb{R}^d$. Then there exists a positive constant C_α, depend only on α, and such that

$$\||D|^\alpha (fo\,\psi) - (|D|^\alpha f)\; o\; \psi\|_{L^2} \leq Ce^{V(t)}V(t)2^{\alpha q}\|f\|_{L^2},$$

with

$$V(t) := \|\nabla v\|_{L^1_t L^\infty}.$$

The following result describes the action of the semi group operator $e^{-t|D|^{\alpha}}$ on distribution, whose Fourier transform is supported in a ring, see [1], [8] and [9].

Proposition 3.2

Let $p \in [1, +\infty]$, and $t, \alpha \in \mathbb{R}_{+}$. Then there exists a positive constant c such that for any $q \in \mathbb{N}$,

$$\left\| e^{-t|D|^{\alpha}} \Delta_q v \right\|_{L^p} \leq c e^{-Ct2^{q\alpha}} \left\| \Delta_q v \right\|_{L^p},$$

where the constants c and C, depend only on the dimension d.

The following can be found in [4] and [7].

Proposition 3.3

Let v be a smooth divergence free vector field. Let also f be a smooth function and θ is a smooth solution of (1.1). Then for every $p \in [1, \infty]$ we have

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau \,.$$

4 Results and discussion

In this section, we prove the main result of the paper.

Theorem 4.1

Let $f \in L^1_{loc}\left(\mathbb{R}_{+}, B^0_{2,1}\right)$ and v be a smooth divergence free vector field of $\mathbb{R}^2$ such that $v \in L^1_{loc}(\mathbb{R}_{+}, Lip(\mathbb{R}^2))$. We consider also a smooth solution θ of the transport-diffusion equation (1.1), with $\alpha = \frac{1}{2}$. Let also $\theta_0 \in B^0_{2,1}$. Then there exists a positive constant c, and such that

$$\|\theta\|_{\tilde{L}^\infty_t B^0_{2,1}} \le ce^{CV(t)}\left(\|\theta_0\|_{B^0_{2,1}} + \|f\|_{L^1_t B^0_{2,1}}\right),$$

where,

$$V(t) := \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

For the proof, we use a new approach based on Lagrangian coordinates combined with paradifferential calculus. The idea of the proof will be done in the spirit of [7] and [12]. First, we prove the smoothing effects for a small interval of time depending of vector v, but it depends not on the initial data.

In the second step, we proceed to division in time thereby extending the estimate at any time arbitrary chosen positive.

4.1 Local estimates

We localize in frequency the evolution equation, and rewriting the equation in Lagrangian coordinates.

Let $q \in \mathbb{N}$, then the Fourier localized function $\Delta_q\theta$ satisfies

$$\Delta_q(\partial_t\theta) + \Delta_q(v.\nabla\theta) + \Delta_q\left(|D|^{\frac{1}{2}} + I\right)\theta = \Delta_q f$$

Using the notation $[\,\Delta_q\,, v.\nabla\,]\theta = \Delta_q(v.\nabla\theta) - v.\nabla\Delta_q\theta$, We get

$$\Delta_q(v.\nabla\theta) = [\,\Delta_q\,, v.\nabla\,]\theta + v.\nabla\,\Delta_q\theta$$

This gives that

$$\partial_t \Delta_q \theta + v.\nabla \Delta_q \theta + [\Delta_q, v.\nabla]\theta + \left(|D|^{\frac{1}{2}} + I\right) \Delta_q \theta = \Delta_q f.$$

Therefore

$$\partial_t \Delta_q \theta + v.\nabla \Delta_q \theta + \left(|D|^{\frac{1}{2}} + I\right) \Delta_q \theta = \Delta_q f - [\Delta_q, v.\nabla]\theta := F_q$$

From Proposition 3.3, we have

$$\|\Delta_q \theta(t)\|_{L^2} + \|\Delta_q \theta(t)\|_{L^2}^2 \leq \|\Delta_q \theta_0\|_{L^2} + \int_0^t \|F_q(\tau)\|_{L^2} \, d\tau.$$

Since we have

$$\|\Delta_q \theta(t)\|_{L^2} \leq \|\Delta_q \theta(t)\|_{L^2} + \|\Delta_q \theta(t)\|_{L^2}^2 \leq \|\Delta_q \theta_0\|_{L^2} + \int_0^t \|F_q(\tau)\|_{L^2} \, d\tau.$$

Therefore

$$\|\Delta_q \theta(t)\|_{L^2} \leq \|\Delta_q \theta_0\|_{L^2} + \int_0^t \|F_q(\tau)\|_{L^2} \, d\tau.$$

Summing over q, yields

$$\sum_q \|\Delta_q \theta(t)\|_{L^2} \leq \sum_q \|\Delta_q \theta_0\|_{L^2} + \int_0^t \sum_q \|F_q(\tau)\|_{L^2} \, d\tau.$$

This gives that,

$$\|\theta\|_{B^0_{2,1}} \leq \|\theta_0\|_{B^0_{2,1}} + \|f\|_{L^1_t B^0_{2,1}} + C \int_0^t \sum_q \|[\Delta_q, v.\nabla]\theta(\tau)\|_{L^2} \, d\tau$$

Therefore, by using Lemma 3.3, we get

$$\|\theta\|_{\tilde{L}^\infty_t B^0_{2,1}} \leq \|\theta_0\|_{B^0_{2,1}} + \|f\|_{L^1_t B^0_{2,1}} + C \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\theta\|_{\tilde{L}^\infty_\tau B^0_{2,1}} d\tau.$$

Using Lemma 2.2.2, to obtain

$$\|\theta\|_{\tilde{L}_t^\infty B_{2,1}^0} \leq C\left(\|\theta_0\|_{B_{2,1}^0} + \|f\|_{L_t^1 B_{2,1}^0}\right) e^{c\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$

Let us now introduce the flow ψ_q of the regularized velocity v,

$$\psi_q(t,x) = x + \int_0^t v\left(\tau, \psi_q(\tau,x)\right) d\tau.$$

We set

$$\bar{\theta}_q(t,x) = \Delta_q\theta(t,\psi_q(t,x)) \qquad \text{and} \qquad \bar{F}_q(t,x) = F_q\left(t,\psi_q(t,x)\right).$$

Then we have,

$$\partial_t\bar{\theta}_q + \left(|D|^{\frac{1}{2}} + I\right)\bar{\theta}_q = \bar{F}_q + \left(|D|^{\frac{1}{2}} + I\right)\left(\Delta_q\theta \ o\ \psi_q\right)$$
$$-\left(\left(|D|^{\frac{1}{2}} + I\right)\Delta_q\theta\right) o\psi_q := \bar{F}_q^1 \qquad (4.1)$$

Since the flow preserves Lebesgue measure, then we obtain

$$\left\|\bar{F}_q\right\|_{L^2} \leq \left\|\Delta_q f\right\|_{L^2} + \left\|[\Delta_q, v.\nabla]\theta\right\|_{L^2} \qquad (4.2)$$

Using now Proposition 3.1, we find that for $q \in \mathbb{Z}$

$$\left\||D|^{\frac{1}{2}}(f o\ \psi) - \left(|D|^{\frac{1}{2}} f\right) o\ \psi\right\|_{L^2} \leq Ce^{CV(t)}V(t)2^{\frac{q}{2}}\|f\|_{L^2}, \quad V(t) := \|\nabla v\|_{L_t^1 L^\infty}.$$

This gives that,

$$\left\|\left(|D|^{\frac{1}{2}} + I\right)\left(\Delta_q\theta \ o\ \psi_q\right) - \left(\left(|D|^{\frac{1}{2}} + I\right)\Delta_q\theta\right) o\ \psi_q\right\|_{L^2}$$
$$\leq Ce^{CV(t)}V(t)2^{\frac{q}{2}}\left\|\Delta_q\theta\right\|_{L^2} \qquad (4.3)$$

Combining (4.2) and (4.3), we obtain

$$\left\|\bar{F}_q^1\right\|_{L^2} \leq \left\|\Delta_q f\right\|_{L^2} + \left\|\left[\Delta_q, v.\nabla\right]\theta\right\|_{L^2} + Ce^{CV(t)}V(t)2^{\frac{q}{2}}\left\|\Delta_q\theta\right\|_{L^2}$$

$$\leq \left\|\Delta_q f\right\|_{L^2} + \left\|\left[\Delta_q, v.\nabla\right]\theta\right\|_{L^2} + Ce^{CV(t)}2^{\frac{q}{2}}\left\|\Delta_q\theta\right\|_{L^2}.$$

Now, we will again localize in frequency the equation (4.1) through the operator Δ_j , $j \in \mathbb{Z}$

$$\partial_t\Delta_j\,\bar{\theta}_q + \left(|D|^{\frac{1}{2}} + I\right)\Delta_j\,\bar{\theta}_q = \Delta_j\bar{F}_q^1,$$

where

$$\Delta_j\bar{F}_q^1 \coloneqq \Delta_j\bar{F}_q + \Delta_j\left(\left(|D|^{\frac{1}{2}} + I\right)\left(\Delta_q\theta\ o\ \psi_q\right) - \left(\left(|D|^{\frac{1}{2}} + I\right)\Delta_q\theta\right)\ o\psi_q\right).$$

Using Duhamel formula,

$$\Delta_j\,\bar{\theta}_q(t,x) = e^{-t\left(|D|^{\frac{1}{2}}+I\right)}\Delta_j\,\theta_q^0 + \int_0^t e^{-(t-\tau)\left(|D|^{\frac{1}{2}}+I\right)}\Delta_j\bar{F}_q(\tau)d\tau$$

$$+\int_0^t e^{-(t-\tau)\left(|D|^{\frac{1}{2}}+I\right)}\Delta_j\left(\left(|D|^{\frac{1}{2}} + I\right)\left(\Delta_q\theta\ o\ \psi_q\right)\right.$$

$$\left.-\left(\left(|D|^{\frac{1}{2}} + I\right)\Delta_q\theta\right)\ o\ \psi_q\right)d\tau$$

Taking the L^2 norm, of the above equality, we get

$$\left\|\Delta_j\bar{\theta}_q(t)\right\|_{L^2} \leq \left\|e^{-t\left(|D|^{\frac{1}{2}}+1\right)}\Delta_j\theta_q^0\right\|_{L^2} + \int_0^t\left\|e^{-(t-\tau)\left(|D|^{\frac{1}{2}}+1\right)}\Delta_j\bar{F}_q(\tau)\right\|_{L^2}d\tau$$

$$+\int_0^t \left\| e^{-(t-\tau)\left(|D|^{\frac{1}{2}}+1\right)} \Delta_j \left(\left(|D|^{\frac{1}{2}} + I \right) (\Delta_q \theta \ o \ \psi_q) - \left(\left(|D|^{\frac{1}{2}} + I \right) \Delta_q \theta \right) o \ \psi_q \right) \right\|_{L^2} d\tau.$$

Using Holder inequality 2.2.1 and Proposition 3.2, we find

$$\|\Delta_j \bar{\theta}_q(t)\|_{L^2} \leq Ce^{-ct2^{\frac{j}{2}}} \|\Delta_j \theta_q^0\|_{L^2} + C\int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \|\Delta_q f(\tau)\|_{L^2} d\tau$$

$$+ C\int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \|[\Delta_q, v.\nabla]\theta(\tau)\|_{L^2} d\tau$$

$$+C\, e^{CV(t)} 2^{\frac{j}{2}} \int_0^t e^{-c(t-\tau)2^{\frac{j}{2}}} \|\Delta_q \theta(\tau)\|_{L^2} d\tau$$

Therefore,

$$\|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^2} \leq C\left(\|\Delta_j \theta_q^0\|_{L^2} + \|\Delta_q f\|_{L_t^1 L^2} \right) + \|[\Delta_q, v.\nabla]\theta\|_{L_t^1 L^2}$$

$$+ Ce^{CV(t)} 2^{\frac{j}{2}} \|\Delta_q \theta\|_{L_t^\infty L^2} \qquad (4.4)$$

Let $N \in \mathbb{N}$ be a fixed number that will be chosen later, and since the flow ψ preserves Lebesgue measure then we write

$$\|\Delta_q \theta\|_{L_t^\infty L^2} = \|\bar{\theta}_q\|_{L_t^\infty L^2}$$

$$\leq \sum_{|j-q|\geq N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^2}$$

$$+ \sum_{|j-q|<N} \|\Delta_j \bar{\theta}_q\|_{L_t^\infty L^2}$$

$$:= I + II.$$

If $j-q\geq N$, it follows from (4.4) and by using Proposition 3.2 that,

$$\|\Delta_j\bar{\theta}_q\|_{L_t^\infty L^2}\leq C\,2^{-|j-q|}e^{\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau}\|\Delta_q\theta\|_{L_t^\infty L^2}$$
$$\leq C\,2^{-|j-q|}e^{V(t)}\,\|\Delta_q\theta\|_{L_t^\infty L^2}.$$

Therefore, we get

$$I\leq C\,2^{-N}e^{CV(t)}\|\Delta_q\theta\|_{L_t^\infty L^2}\qquad(4.5)$$

For the terme II,

$$II=\sum_{|j-q|<N}\|\Delta_j\bar{\theta}_q\|_{L_t^\infty L^2},$$

we use (4.4), to get

$$II\leq C\,\|\Delta_q\theta_0\|_{L^2}+\|\Delta_q f\|_{L_t^1L^2}+\|[\Delta_q,v.\nabla]\theta\|_{L_t^1L^2}+$$
$$Ce^{CV(t)}2^{\frac{j}{2}}\,\|\Delta_q\theta\|_{L_t^\infty L^2}\quad(4.6)$$

This gives in view of (4.5) and (4.6), that

$$\|\Delta_q\theta\|_{L_t^\infty L^2}\leq I+II.$$

Thus

$$\|\Delta_q\theta\|_{L_t^\infty L^2}\lesssim 2^{-N}e^{CV(t)}\|\Delta_q\theta\|_{L_t^\infty L^2}+\|\Delta_q\theta_0\|_{L^2}$$
$$+\|\Delta_q f\|_{L_t^1L^2}+\|[\Delta_q,v.\nabla]\theta\|_{L_t^1L^2}+e^{CV(t)}2^{\frac{j}{2}}\|\Delta_q\theta\|_{L_t^\infty L^2}$$

Therefore,

$$\|\Delta_q\theta\|_{L_t^\infty L^2}\lesssim\|\Delta_q\theta_0\|_{L^p}+\left[2^{-N}+2^{\frac{j}{2}}\right]e^{CV(t)}\,\|\Delta_q\theta\|_{L_t^\infty L^2}$$
$$+\|\Delta_q f\|_{L_t^1L^2}+\int_0^t\|[\Delta_q,v.\nabla]\theta(\tau)\|_{L^2}d\tau.$$

Now, we claim that there exist two constants $N \in \mathbb{N}$ and C_1 such that if $V(t) \leq C_1$, then

$$2^{-N} + 2^{\frac{j}{2}} \leq \frac{1}{2C}.$$

To show this, we take first t such that $V(t) \leq 1$, which is possible since $\lim_{t\to 0^+} V(t) = 0$. Second, we choose N in order to get $2^{-N} + 2^{\frac{j}{2}} \leq \frac{1}{2C}$. By taking again $V(t)$ sufficiently small, then under this assumption $V(t) \leq C_1$, we obtain for $q \geq -1$,

$$\left\|\Delta_q \theta\right\|_{L_t^\infty L^2} \leq \left\|\Delta_q \theta_0\right\|_{L^2} + \left\|\Delta_q f\right\|_{L_t^1 L^2} + \int_0^t \left\|\left[\Delta_q, v.\nabla\right]\theta(\tau)\right\|_{L^2} d\tau \quad (4.7)$$

summing over q, and using Lemma 3.3 and for $V(t) \leq C_1$, we find

$$\|\theta\|_{\tilde{L}_t^\infty B_{2,1}^0} \leq C\|\theta_0\|_{B_{2,1}^0} + C\|f\|_{L_t^1 B_{2,1}^0} + C\int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\theta\|_{\tilde{L}_t^\infty B_{2,1}^0} d\tau.$$

Using Lemma 2.2.2, yields

$$\|\theta\|_{\tilde{L}_t^\infty B_{2,1}^0} \leq C\left(\|\theta_0\|_{B_{2,1}^0} + \|f\|_{L_t^1 B_{2,1}^0}\right) e^{c\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} \qquad (4.8)$$

Therefore, the result is proved for small time.

4.1.2 Globalization

Let us now see how to extend this for an arbitrary positive time T. We take a partition $(T_i)_{i=0}^N$ of the interval, $[0, T]$ such that

$$\int_{T_i}^{T_{i+1}} \|\nabla v(\tau)\|_{L^\infty} d\tau \approx C_0, \forall\, i \in [0, N]..$$

Reproducing the same argument of (4.8), we obtain

$$\|\theta\|_{\tilde{L}^\infty_{[T_i, T_{i+1}]} B_{2,1}^0} \leq C\|\theta(T_i)\|_{B_{2,1}^0} + \int_{T_i}^{T_{i+1}} \|f(\tau)\|_{B_{2,1}^0} d\tau.$$

Summing these estimates on $i = 1$, to $i = N$, and using triangle inequality, gives

$$\|\theta\|_{\tilde{L}_T^\infty B_{2,1}^0} \leq C \sum_{i=0}^{N-1} \|\theta(T_i)\|_{B_{2,1}^0} + C \int_0^T \|f(T)\|_{B_{2,1}^0} \, dT.$$

Thus

$$\|\theta\|_{\tilde{L}_T^\infty B_{2,1}^0} \leq CN \left(\|\theta_0\|_{B_{2,1}^0} + \|f\|_{L_T^1 B_{2,1}^0} \right) e^{CV(T)}.$$

It suffices to choose N such that $CN \approx V(t)$, then

$$\|\theta\|_{\tilde{L}_T^\infty B_{2,1}^0} \leq V(t) \left(\|\theta_0\|_{B_{2,1}^0} + \|f\|_{L_T^1 B_{2,1}^0} \right) e^{CV(T)}.$$

Therefore, we get

$$\|\theta\|_{\tilde{L}_T^\infty B_{2,1}^0} \leq C e^{CV(T)} \left(\|\theta_0\|_{B_{2,1}^0} + \|f\|_{L_T^1 B_{2,1}^0} \right).$$

This is the desired result, and the proof of the theorem is now complete.

5 Conclusion

We have proved a smoothing effect for a transport diffusion equation in space dimension two. For this, we used the concept of the flow associated to the velocity and a new approach based on Lagrangian coordinates combined with paradifferential calculus.

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