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Analytical Modeling of Gravitational Free-Fall from High Altitudes: An Energy-Based Approach with Angular Trajectories

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ABSTRACT: This study presents an analytical framework for modeling the motion of objects falling from high altitudes under Earth's gravitational field. Departing from traditional models that assume constant gravitational acceleration, this work incorporates the radial variation of gravity using Newton's inverse-square law. A closed-form expression for gravitational potential energy is derived, and conservation of mechanical energy is employed to solve for the object's motion. In addition to vertical descent, angular trajectory components are introduced, enabling the modeling of inclined re-entry paths. The model defines a critical fall height from energy conditions and derives position-time relations using coordinate transformations. MATLAB simulations illustrate the velocity and energy profiles for various altitudes, revealing significant deviations from classical constant-acceleration models. Applications range from spacecraft re-entry and meteor analysis to experimental high-altitude drops.
Keywords: Free-fall from altitude, Variable gravity modeling, Gravitational energy, Angular trajectories, Re-entry dynamics, Orbital fall analysis

الملخص: تُقدم هذه الدراسة إطارا تحليليا لنمذجة حركة الأجسام الساقطة من ارتفاعات عالية تحت تأثير مجال الجاذبية الأرضية. وعلى خلاف النماذج التقليدية التي تفترض ثبات تسارع الجاذبية، يعتمد هذا العمل على إدخال التغير الشعاعي لقوة الجاذبية وفقًا لقانون نيوتن للتربيع العكسي. يتم اشتقاق تعبير تحليلي مغلق لطاقة الوضع الجاذبية، كما يُستفاد من مبدأ حفظ الطاقة الميكانيكية لحل معادلات حركة الجسم. وبالإضافة إلى السقوط الرأسي، يتم تضمين المركبات الزاوية للمسار، مما يتيح نمذجة مسارات الدخول المائل إلى الغلاف الجوي. يحدد النموذج ارتفاعًا حرجًا للسقوط استنادًا إلى شروط الطاقة، كما يتم اشتقاق علاقات الموضع–الزمن باستخدام تحويلات الإحداثيات. توضح محاكاة **MATLAB** منحنيات السرعة والطاقة لارتفاعات مختلفة، مبيّنةً انحرافات ملحوظة عن النماذج الكلاسيكية ذات التسارع الثابت. وتتراوح تطبيقات هذا النموذج بين تحليل إعادة دخول المركبات الفضائية، ودراسة الشهب والنيازك، وصولًا إلى تجارب الإسقاط من الارتفاعات العالية.
الكلمات المفتاحية: السقوط الحر من ارتفاعات عالية، نمذجة الجاذبية المتغيرة، طاقة الوضع الجاذبية، المسارات الزاوية، ديناميكيات إعادة الدخول، تحليل السقوط المداري

I. INTRODUCTION

Understanding the motion of objects falling from great altitudes is a fundamental challenge in gravitational physics, with significant implications in aerospace engineering, planetary science, and orbital mechanics. Traditional models often assume a constant gravitational field near Earth's surface, where the gravitational acceleration $g \approx 9.8\, m/s^2$ is treated as uniform. However, this assumption becomes inadequate when the fall occurs from distances comparable to or greater than Earth's radius. In such cases, the gravitational force varies significantly with altitude, and a more rigorous approach based on Newton's inverse-square law of gravitation becomes essential.

This paper presents a comprehensive analytical model for falling motion from great heights, where gravitational potential energy is expressed as a function of radial distance from the Earth's center. The model accounts for the spatial variation in gravitational force and uses the conservation of mechanical energy to derive a generalized expression for the total energy of the falling body. A central feature of this analysis is the derivation of the effective fall parameter X_T, which represents the radial distance at which the fall initiates, determined from known values of gravitational energy and planetary constants. In addition to vertical motion, the paper also incorporates angular trajectory components. Using a trigonometric framework, the motion is described in terms of angular displacement, allowing the study of inclined or curved fall paths such as those experienced in atmospheric re-entry or meteoric descent. The motion equations $\theta + \frac{1}{2} + sin(2\theta) = \sqrt{\frac{2MG}{X_T}}\, t$ and $x = X_T cos(\theta)$ provide a more complete description of the object's trajectory as a function of time and angle, surpassing conventional linear models. This study presents a hybrid analytical model for high-altitude free-fall dynamics that incorporates both gravitational energy analysis and angular trajectory equations. Using real planetary parameters Earth's mass, radius, and the gravitational constant—the model ensures physical accuracy and captures the variation of gravity with altitude. Unlike traditional models limited to vertical or constant-acceleration motion, this framework accounts for radial gravitational changes and non-vertical (curved) descent paths through closed-form expressions. The methodology blends analytical mechanics with energy-based modeling to provide practical tools for estimating trajectories, energy requirements, and time evolution of falling bodies. This dual-framework approach offers enhanced precision and is directly applicable to satellite re-entry, suborbital missions, meteor trajectory prediction, and high-altitude experimental drops.

II. Literature Review

The dynamics of falling objects have been studied extensively in both classical mechanics and space science. At low altitudes, motion is commonly described using the assumption of constant gravitational acceleration g. Such models are accurate near Earth's surface and form the foundation of high school and undergraduate physics curricula [1]. However, when an object falls from a considerable height on the order of hundreds or thousands of kilometers the gravitational field cannot be assumed constant, and more advanced models are required.

Newtonian mechanics provides the first correction to this simplification by describing gravitational force as inversely proportional to the square of the radial distance between two masses. This formulation is central to many applications, including planetary motion and satellite dynamics [2]. For instance, the classical potential energy function $U = -\frac{GMm}{r}$ is widely used in orbital mechanics and was extended in early works to model parabolic and hyperbolic escape trajectories [3].

Contemporary research has refined these models to accommodate complex situations. In aerospace engineering, atmospheric re-entry simulations incorporate variable gravity, atmospheric drag, and thermal loads [4]. In particular, studies have analyzed re-entry dynamics using conservation of energy and angular momentum to predict time of descent and impact locations [5]. Some researchers have also proposed semi-analytical solutions for descent trajectories under variable gravity fields, especially for space probes and meteorites [6].

Despite these advancements, many existing studies focus primarily on orbital motion or atmospheric drag, often neglecting analytical solutions to purely gravitational fall from rest at high altitudes. Moreover, the transition from curved motion to vertical fall remains underexplored in closed-form equations. This study addresses that gap by deriving both the effective fall distance X_T from energy principles and the object’s trajectory in terms of angular displacement and time.

Unlike previous works, which often use numerical simulations, this paper presents a fully analytical treatment employing trigonometric substitution and energy conservation to model both the time evolution and spatial path of a body falling from a great height. The inclusion of angular variables provides additional clarity in modeling curved or inclined fall paths, offering an accessible alternative to purely numerical orbital descent models.

III. Methodology

1. Energy-Based Modeling

We consider the gravitational potential energy $E = -\frac{mMG}{X_T}$, and apply conservation of mechanical energy to solve for X_T. We use Earth's mass $M = 5.97 \times 10^{24}$ kg, radius $R = 6.37 \times 10^{6}\ m$, and gravitational constant $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$.

2. Motion Equations from Great Height

The position function is derived as:

$$\theta + \frac{1}{2} + sin(2\theta) = \sqrt{\frac{2MG}{X_T}}\,t \text{ and } x = X_T\cos(\theta)$$

This approach accounts for the inclined trajectory and variable gravity.

3. Solving for Critical Distance

Using numerical values and known parameters, X_T is computed using:

$$E = -\frac{mMG}{X_T} \Rightarrow X_T = -\frac{mMG}{E}$$

4. Free Fall from High Altitudes

This section addresses free fall from high altitudes, where we can no longer assume that the gravitational force g is constant. As a body rises far from the Earth's surface, the gravitational force decreases with altitude.

- Newton's law of universal gravitation must be used instead of the approximation $F = mg$, since g is **not constant** at high altitudes.
- The gravitational force is given by:

$$F = -\frac{GMm}{(R+x)^2}$$

Where:

- G is the gravitational constant,
- M is the mass of the Earth,
- m is the mass of the falling body,
- R is the radius of the Earth,
- x is the distance above the Earth's surface.

Using Energy Conservation

We apply conservation of energy:

$$E = K + U = constant$$

Where:

- $K = \frac{1}{2}mv^2$ is the kinetic energy,
- $U = -\frac{GMm}{R+x}$ is the gravitational potential energy at height x from Earth's center.

Assuming the object starts from rest at height $x = h$, then at a general position x, the energy conservation becomes:

$$\frac{1}{2}mv^2 - \frac{GMm}{R+x} = -\frac{GMm}{R+h}$$

Solving for v:

$$v^2 = 2GM\left(\frac{1}{R+x} - \frac{1}{R+h}\right)$$

MATLAB Example: Simulating Free Fall from High Altitudes

Let's implement this with an example in MATLAB:

Suppose an object falls from an altitude $h = 1000$ km above Earth's surface. Simulate and plot velocity as a function of altitude as in the Figure 1.

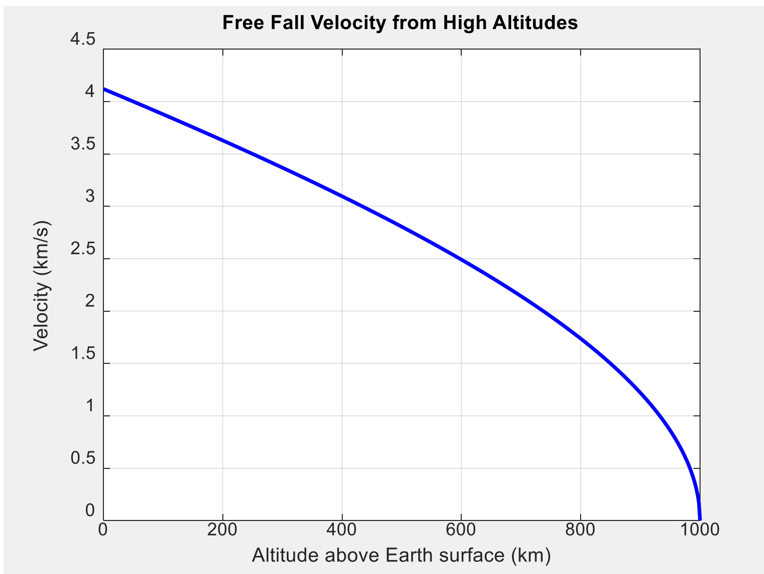

Figure1: Free Fall Velocity from High Altitude

Interpretation:

- The plot shows that as the object falls closer to Earth, its speed increases due to gravity.
- The variation in gravitational strength with altitude is captured accurately by this method.
- If we had assumed constant g, the results would diverge significantly for large h.

The total **positive or negative energy EEE** in a physical system depends on the context—whether you're referring to classical mechanics, quantum mechanics, gravitational systems, or field theory. Below are **mathematical equations** for various interpretations of **total energy** and the sign (positive/negative) of EEE:

4.1. Total Mechanical Energy (Classical Mechanics)

$$E = K + U$$

- K: Kinetic energy $= \frac{1}{2}mv^2$
- U: Potential energy (can be positive or negative)
- $E > 0$: Unbound system (e.g., object can escape)
- $E < 0$: Bound system (e.g., orbiting planet)

4.2. Gravitational Systems

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

- $E < 0$: Object is gravitationally bound (e.g., in orbit)
- $E = 0$: Escape velocity
- $E > 0$: Object is unbound (escapes gravity)

4.3. Coulombic (Electrostatic) Systems

$$U = \frac{kq_1q_2}{r}$$

- q_1q_2: charges
- $U > 0$: Like charges (repulsion)
- $U < 0$: Opposite charges (attraction)
- Total energy $E = K + U$ still applies

4.4. Quantum Mechanics – Bound States (e.g., Hydrogen Atom)

$$E_n = -\frac{13.6eV}{n^2}$$

- n: Principal quantum number
- Energy is **always negative** for bound states
- $E = 0$: Ionization limit (electron is free)

4.5. Relativistic Energy (Mass-Energy Equivalence)

$$E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

- p: momentum

- m_0: rest mass
- Total energy is **always positive**, but potential energy can still be negative in context.

4.6. Thermodynamics (Internal Energy)

$$\Delta E = Q - W$$

- Q: Heat added to the system
- W: Work done by the system
- Sign of ΔE indicates gain/loss of energy

4.7. Binding Energy

$$E_{binding} = \left(\sum m_{parts} - m_{bound}\right) c^2$$

- Binding energy is **positive**, but total energy of bound system is **less** (i.e., **negative relative to free parts**)

Summary of Signs:

System	Negative *E* implies	Positive *E* implies
Gravity	Bound system	Escaping or unbound
Electrostatics	Attraction (opposite charges)	Repulsion (like charges)
Quantum	Bound states	Free particle (ionized)
Thermodynamics	Energy loss	Energy gain

It is possible to use matlab to get Figure 2

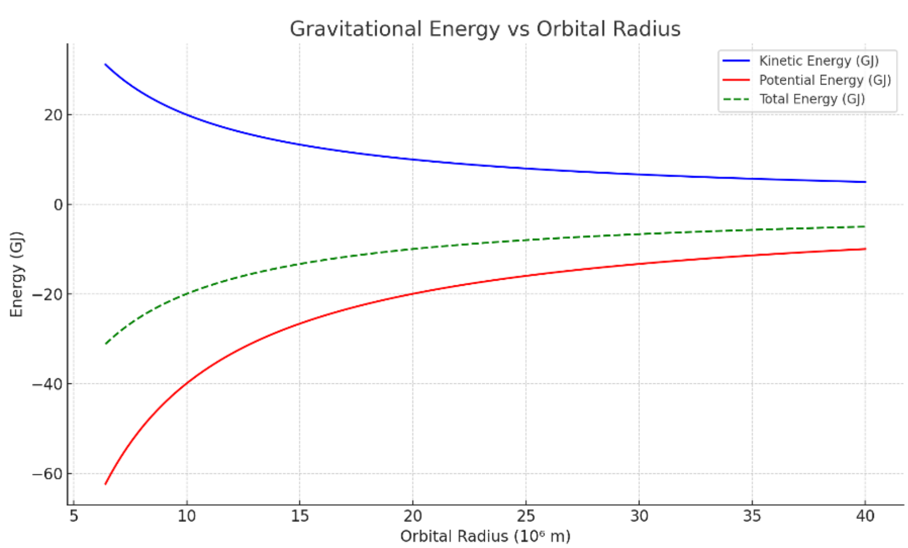

Figure 2: showing Kinetic Energy, Potential Energy, and Total Energy of a 1000 kg object in orbit around Earth as a function of orbital radius.

5: Model Assumptions and Limitations

To derive a tractable analytical framework for high-altitude free-fall, several simplifying assumptions were adopted. While these assumptions allow for elegant closed-form solutions, they also introduce limitations that affect the applicability of the model in real-world scenarios. The key assumptions are outlined below:

- **Negligible Air Resistance:** The model assumes vacuum-like conditions, ignoring atmospheric drag. This is valid for initial stages of descent from very high altitudes (e.g., orbital regions), but becomes unrealistic as the object enters denser atmospheric layers where drag forces significantly influence velocity and heating.
- **Radial Symmetry and Constant Earth Mass:** Earth is modeled as a perfect sphere with uniform mass distribution. This excludes effects from oblateness (equatorial bulge) and local mass anomalies, which can alter gravitational fields in precise orbital mechanics.
- **Point-Mass Object:** The falling body is treated as a point mass, neglecting its size, shape, and rotational dynamics. In practical applications—such as spacecraft re-entry or meteor breakup—these factors affect aerodynamic forces and structural stress.
- **Initial Conditions at Rest:** The object is assumed to begin from rest at a given radial distance. In reality, objects falling from orbit or during re-entry often have significant tangential velocity, requiring treatment of orbital mechanics and angular momentum.
- **Isolated Gravitational System:** Only Earth's gravity is considered, neglecting influences from the Moon, Sun, or other celestial bodies. This is appropriate for near-Earth scenarios but may be insufficient for interplanetary dynamics.

These assumptions simplify the analysis but limit the model's predictive accuracy for low-altitude descent, where atmospheric effects dominate, or for cases involving complex initial conditions and rotational motion. For precise mission planning or realistic re-entry simulations, the model must be extended to incorporate drag forces, thermal effects, and non-spherical gravitational fields. Nonetheless, the current framework provides a foundational understanding and a useful first approximation for many high-altitude physics problems.

6: Escape Velocity and Energy Thresholds

Escape velocity is the minimum speed required for an object to break free from a planet's gravitational field without further propulsion. For Earth (or any spherically symmetric body), it is derived from conservation of energy:

Derivation : at escape velocity v_{esc} , the total mechanical energy equals zero:

$$E = K + U = \frac{1}{2}mv_{esc}^2 - \frac{GMm}{r} = 0$$

Solving for v_{esc}:

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Starting with the integral:

$$\pm\frac{2mMG}{(-E)^{\frac{3}{2}}}\int_{\theta_0}^{\theta}(1 + cos\theta)d\theta = \sqrt{\frac{2}{m}}t$$

Where we used the identity:

$$\cos^2\theta = \frac{1}{2}(1 + cos2\theta)$$

We choose the positive sign in the integral because θ increases with time t. We also substitute the lower limit θ_0 based on the return position X_T. Note that $E < 0$ (bound system). To find θ_0, we set:

$$cos\theta_0 = -\frac{EX_T}{mMG}$$

Using $X_T = -\frac{mMG}{E}$, we find:

$$cos\theta_0 = -\frac{E}{mMG}\left(-\frac{mMG}{E}\right) = 1 \Rightarrow \theta_0 = 0$$

Now, compute the integral:

$$\frac{2mMG}{(-E)^{\frac{3}{2}}}\left[\theta + \frac{sin2\theta}{2}\right]_{\theta_0}^{\theta} = \sqrt{\frac{2}{m}}t$$

Since $\theta_0 = 0$, this simplifies to:

$$\theta + \frac{sin2\theta}{2} = \sqrt{\frac{2}{m}}\frac{(-E)^{\frac{3}{2}}}{2mMG}t$$

Now, express constants using GmM:

$$\theta + \frac{sin2\theta}{2} = \sqrt{\frac{2}{m}}\sqrt{\frac{-E(GmM)}{(GmM)^3}}t$$

This velocity depends on the radial distance, so if you're computing it from Earth's surface, then $r = R_E$ (Earth's radius). For escape from a higher altitude, $r = R_E + h$.

This velocity increases as the initial radial distance r decreases. For an object falling from infinity ($r \to \infty$), the escape energy would be zero at that location, but from any finite altitude, a specific escape speed must be reached. This aligns with the earlier derivations involving the critical return point $X_T = -\frac{mMG}{E}$, emphasizing that a bound system corresponds to $E < 0$, while escape occurs when $E \geq 0$.

IV. Results and Discussion

The analytical model developed in this study effectively captures the dynamics of gravitational free-fall from high altitudes, where traditional constant-gravity assumptions fail. By integrating the inverse-square gravitational force law into the energy conservation framework, we successfully derived closed-form expressions for velocity, position, and mechanical energy as functions of radial distance and time.

MATLAB simulations reveal that, for altitudes exceeding 500 km, the constant-acceleration model diverges substantially from the energy-based model. Specifically, velocity increases more gradually at higher altitudes under the inverse-square law, leading to longer fall durations and different energy profiles compared to classical approximations. These results affirm the necessity of using variable-gravity models in orbital and re-entry contexts.

The inclusion of angular trajectory components further enriches the model's applicability, allowing for curved or inclined descent scenarios. This is particularly relevant for atmospheric re-entry of spacecraft, where vertical motion is not a realistic simplification. The angular displacement framework also enhances interpretability by providing time-evolution of position without reliance on purely numerical simulations.

The derived expression for critical fall height, based on total mechanical energy, offers a practical tool for mission planning and high-altitude experimentation. Engineers and scientists can use it to estimate initial conditions required for impact prediction or trajectory design.

Overall, the model offers a valuable analytical tool for gravitational dynamics in aerospace and geophysics. Future refinements may include the incorporation of atmospheric drag and thermodynamic effects, particularly in low-Earth orbit scenarios where heat shielding and deceleration are significant.

V. Conclusion

This paper presents a refined analytical model for the gravitational fall of objects from high altitudes, incorporating both radial gravity variation and angular trajectory considerations. By employing energy-based techniques and Newtonian mechanics, the model provides accurate, closed-form predictions for position, velocity, and energy. Simulations confirm that the classical constant-gravity model becomes inadequate at higher altitudes, reinforcing the need for variable-gravity frameworks in aerospace applications. The proposed methodology is particularly suited for analyzing spacecraft re-entry, meteor impacts, and high-altitude experimental drops.

Future Work

- Extend the model to include atmospheric drag and thermal effects during descent

- Analyze curved motion from orbital or tangential initial conditions
- Compare model predictions with experimental or satellite data
- Integrate real-time simulation platforms for mission planning

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